

This appendix is a brief review of the stability theory by the Lyapunov method for a nonlinear autonomous system

$$\dot{x} = f(x), \tag{A4.1}$$

where x is an n -dimensional state vector and $f(x)$ is a nonlinear function of x . In many cases the dynamics of a system are given not by equation A4.1 but by an n th-order differential equation

$$x^{(n)} = f(x, \dot{x}, \dots, x^{(n-1)}), \tag{A4.2}$$

where x is a scalar variable and $x^{(i)}$ is its i th time derivative. Even in that case, however, we can reduce the expression A4.2 into A4.1 by letting

$$x = [x, \dot{x}, \ddot{x}, \dots, x^{(n-1)}]^T \tag{A4.3}$$

and

$$f(x) = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(n-1)} \\ f(x, \dot{x}, \dots, x^{(n-1)}) \end{bmatrix}. \tag{A4.4}$$

We can reduce the expression A4.2 to A4.1. Thus, A4.1 is a fairly general expression for dynamics systems. In the following, we discuss the stability of the equilibrium point of system A4.1, that is, the equilibrium state \hat{x} such that

$$f(\hat{x}) = 0. \tag{A4.5}$$

Note, however, that there is no loss of generality even if we assume that the equilibrium point is at the origin 0 , i.e., $\hat{x} = 0$. We can say this because when $\hat{x} \neq 0$ we can always transfer the equilibrium point to 0 by letting $x - \hat{x}$ be a new state vector x and expressing equation A4.1 in terms of the new state vector.

The stability of the origin 0 is defined as follows:

- (i) The origin 0 is said to be stable if there exists a $\delta > 0$ for any given $\varepsilon > 0$ such that the solution $x(t)$ to equation A4.1 with an arbitrary initial state $x(0)$ such that $\|x(0)\| < \delta$ satisfies $\|x(t)\| < \varepsilon$ for all $t \geq 0$.
- (ii) The origin 0 is said to be asymptotically stable if it is stable and there

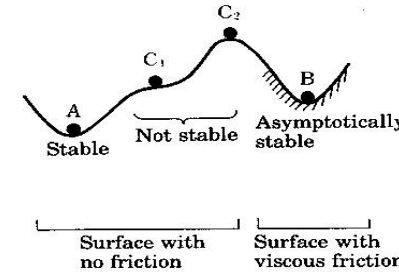


Figure A4.1 Stability of equilibrium point.

exists a $\delta' > 0$ such that the solution $x(t)$ starting from any initial state $x(0)$ satisfying $\|x(0)\| < \delta'$ converges to 0 as time goes to infinity.

(iii) The origin 0 is globally asymptotically stable if it is stable and the solution $x(t)$ starting from any initial point $x(0)$ converges to 0 as time goes to infinity.

For example, consider the ball in figure A4.1 sliding on a curved surface with gravity acting downward. The equilibrium point A is stable (the ball continues to oscillate near the bottom because of the frictionless surface), B is asymptotically stable (the ball reaches the bottom and stops there after infinite length of time because of viscous friction), and C_1 and C_2 are not stable.

Before defining the Lyapunov function, we introduce the concept of positive-definite and positive-semidefinite functions of x . Let us consider a scalar function $v(x)$ defined over a region Ω including the origin. The function $v(x)$ is said to be positive definite (or positive semidefinite) in Ω if $v(0) = 0$ and $v(x) > 0$ (or $v(x) \geq 0$) for any $x \in \Omega$ such that $x \neq 0$. Similarly, $v(x)$ is said to be negative definite (or negative semidefinite) in Ω if $v(0) = 0$ and $v(x) < 0$ (or $v(x) \leq 0$) for any $x \in \Omega$ such that $x \neq 0$.

The function $v(x)$ is called the *Lyapunov function* if $v(x)$ is positive definite in Ω , its partial derivative $\partial v(x)/\partial x$ is continuous, and the time derivative

$$\dot{v}(x) = \frac{dv(x(t))}{dt} = \left(\frac{\partial v(x)}{\partial x} \right)^T f(x) \tag{A4.6}$$

is negative semidefinite.

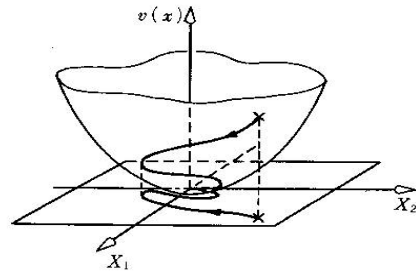


Figure A4.2
Lyapunov function.

Theorem 1 The equilibrium point $\mathbf{0}$ of system A4.1 is stable if and only if the following condition is satisfied in a neighborhood Ω of the origin:

(i) There exists a Lyapunov function.

The equilibrium point $\mathbf{0}$ is asymptotically stable if and only if condition i and the following condition are satisfied in Ω :

(ii) The time derivative of $v(x)$, $\dot{v}(x)$, is negative definite.

Condition ii can be replaced by the following:

(ii') No solution $x(t)$ to equation A4.1 for any initial condition $x(0)$ such that $x(0) \neq \mathbf{0}$ satisfies $\dot{v}(x(t)) = 0$ identically. \square

This theorem is called *Lyapunov's stability theorem*.

Schematically, the Lyapunov function for the case of $n = 2$ is like the cup shown in figure A4.2. Condition i in the above theorem implies that the point $[x^T(t), v(x(t))]^T$ on the cup, which corresponds to a point $x(t)$ of the trajectory of the system A4.1 on the X_1 - X_2 plane, does not ascend as time passes. Condition ii implies that the point always descends until the point reaches the origin after an infinite length of time. Condition ii' implies that, as shown in figure A4.3, although the point on the cup may not descend but may move in a horizontal direction either at an instant of time or for a finite duration of time, it eventually starts to descend again.

The following theorem gives a condition for global asymptotic stability.

Theorem 2 The equilibrium point $\mathbf{0}$ of system A4.1 is globally asymptotically stable if and only if conditions i and ii and the following condition

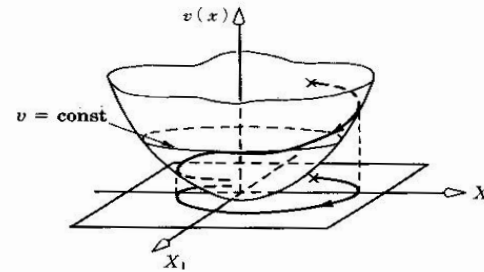


Figure A4.3
Condition ii'.

are satisfied in the whole space of x :

(iii) The function $v(x)$ tends to infinity when $\|x\|$ tends to infinity.

Condition ii can be replaced by condition ii'. \square

The next result is a generalization of condition ii'. A subset S of the n -dimensional Euclidean space R^n of x is called an invariant set for A4.1 if the solution trajectory of A4.1 starting from arbitrary initial state $x(0)$ in S remains in S all the time.

Theorem 3 Let us assume that there exists a Lyapunov function $v(x)$ for system A4.1 in the whole space of x . Let E be the set of all states x satisfying $\dot{v}(x) = 0$, and let M be the largest invariant set for A4.1 included in E . Then an arbitrary finite solution of A4.1 tends to M as time goes to infinity. In addition, if condition iii is satisfied for all x , all solutions of A4.1 tend to M as time goes to infinity. \square

A special case of theorem 3 where M consists only of the origin $\mathbf{0}$ corresponds to the case where condition ii' holds in theorem 2.

As a candidate for a Lyapunov function, we often use a quadratic form $x^T A x$, where A is a real symmetric matrix. When A is not symmetric, the matrix $B = (A + A^T)/2$ is symmetric and satisfies $x^T A x = x^T B x$, so without any loss of generality we can assume that A is symmetric. The following classes of symmetric matrices are useful. An $n \times n$ real symmetric matrix A is said to be positive definite (positive semidefinite) if $x^T A x > 0$ (≥ 0) for any nonzero n -dimensional real vector x . Similarly, A is said to be negative

definite (negative semidefinite) if $-A$ is positive definite (positive semidefinite).

The following theorem is useful for judging whether a given matrix A is positive definite (positive semidefinite) or not. Let the (i, j) element of A be $a(i, j)$, and let the principal minors be denoted by $A(i_1, i_2, \dots, i_k)$:

$$A(i_1, i_2, \dots, i_k) = \begin{vmatrix} a(i_1, i_1) & a(i_1, i_2) & \dots & a(i_1, i_k) \\ a(i_2, i_1) & a(i_2, i_2) & \dots & a(i_2, i_k) \\ \vdots & \vdots & \ddots & \vdots \\ a(i_k, i_1) & a(i_k, i_2) & \dots & a(i_k, i_k) \end{vmatrix}, \quad (\text{A4.7})$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$, $k = 1, 2, \dots, n$, and $|\cdot|$ denotes the determinant.

Theorem 4 (Sylvester's criterion)

(i) A symmetric matrix A is positive definite if and only if all the principal minors are positive, or, equivalently,

$$A(1, 2, \dots, k) > 0, \quad k = 1, 2, \dots, n. \quad (\text{A4.8})$$

(ii) A symmetric matrix A is positive semidefinite if and only if all the principal minors are non-negative:

$$A(i_1, i_2, \dots, i_k) \geq 0, \quad (\text{A4.9})$$

where $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $k = 1, 2, \dots, n$. \square

The reader who wish to learn more about Lyapunov stability theory is referred to J. La Salle and S. Lefschetz, *Stability by Lyapunov's Direct Method* (Academic, 1961).